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A CONCORDANCE CLASSIFICATION OF *PL* HOMEOMORPHISMS OF $S^p \times S^q$

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§1. INTRODUCTION

LET P be a polyhedron and Q a subpolyhedron of P. Let $PL(P; Q)$ be the group of PL homeomorphisms of P keeping Q pointwise fixed. If we take Q to be the empty set, then $PL(P; Q)$ is simply denoted by $PL(P)$. Given two elements f and g of $PL(P; Q)$, we shall say that f is *concordant* (or pseudo-isotopic) to g, if there is an element h of $PL(I \times P; I \times Q)$ such that $h(0, x) = (0, f(x))$ and $h(1, x) = (1, g(x))$ for any point x in P, where $I = [0, 1]$. The element h is called a *concordance* from f to g. Let $TPL(P; Q)$ be the subgroup of $PL(P; Q)$ consisting of elements which are concordant to the identity. Then the subgroup $TPL(P; Q)$ is a normal subgroup of $PL(P; Q)$, see Lemma 2.1. We define $C_{PL}(P; Q) = PL(P; Q)/TPL(P; Q)$. The group $C_{PL}(P; Q)$ will be called the *concordance group* of $PL(P; Q)$. Meanwhile, in our previous paper [6], we have introduced the concept of combinatorial n-prebundles and the structural group PR_n which is an abstract simplicial group complex with homotopy groups $\pi_k(PR_n)$. In this paper we shall show that the order of the group $C_{PL}(S^p \times S^q)$ is expressed by the orders of the groups $\pi_k(PR_n)$ except for the case $p = q =$ odd.

The results are as follows:

THEOREM A. *Suppose that $p > q \geq 2$. Then there is a one-to-one correspondence between the set $C_{PL}(S^p \times S^q)$ and the set $\pi_p(PR_{q+1}) \times \pi_q(PR_{p+1}) \times Z_2 \times Z_2$.*

THEOREM B. *Suppose that p is an even integer. Let G be a subgroup generated by the matrices $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ of the group $GL(2, Z)$ of all non-singular 2×2-matrices over the integral ring Z. Then there is a one-to-one correspondence between the set $C_{PL}(S^p \times S^p)$ and the set $\pi_p(PR_{p+1}) \times \pi_p(PR_{p+1}) \times G$.*

THEOREM C. *Suppose that $p \geq 2$. Then the group $C_{PL}(S^p \times S^1)$ is isomorphic to the direct sum $Z_2 + Z_2 + Z_2$.*

Note. The Theorems above are essentially PL versions of H. Sato's results [10].

Along the way to prove the Theorems we shall show the existence of self-knotted tori.

THEOREM D. *Suppose that $q \geq p + 2$ and $p \equiv 3$ (mod 4). Then $S^p \times S^q$ is self-knotted. That is to say there is a PL homeomorphism of $S^p \times S^q$ which is homotopic to the identity but not isotopic to the identity.*

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However, in case $S^p \times S^q = S^p \times S^1$, we have the following pseudo-unknotted Theorem;

THEOREM E. *Let* $G(S^p \times S^1)$ *be a H-space of homotopy equivalences of* $S^p \times S^1$. *For* $p \geq 2$, *there is a monomorphism* $C_{PL}(S^p \times S^1) \to \pi_0(G(S^p \times S^1))$.

Theorem C has the following implication;

THEOREM F. *For* $p \geq 2$, *there are at most two PL inequivalent PL p-knots with a given exterior, where we mean by a PL p-knot a locally flat* $PL(p+2, p)$*-sphere pair and by an exterior of a PL p-knot the complement of an open regular neighborhood of the embedded PL p-sphere.*

Finally we shall reduce the Hauptvermutung for *PL* knots to the Strong Hurewicz Conjecture for exteriors of *PL* knots; That is,

Conjecture. Let E_1 and E_2 be exteriors of some *PL* p-knots. Given a homotopy equivalence between the pairs (E_1, bE_1) and (E_2, bE_2) then it is homotopic to a *PL* homeomorphism from E_1 to E_2 as a map of pair.

THEOREM G. *Suppose that the Conjecture above is true. Then two PL p-knots are PL equivalent if and only if they are topologically equivalent.*

We shall call a *PL* knot to be *abelian* if its exterior has an abelian fundamental group, necessarily, isomorphic to the infinite cyclic group Z.

At present, by virtue of Browder's result, [2], Corallary 2.4, we may deduce the following;

THEOREM H. *Suppose that* $p \geq 5$. *Then there are at most two PL inequivalent classes in topologically equivalent classes of abelian PL p-knots whose exteriors have finitely generated homotopy groups.*

(Compare the result of [11].)

ADDED IN PROOF. By C. T. C. Wall, "Surgery of compact manifolds," extending Browder-Novikov's technique, the conjecture above is true at least for abelian p-knots ($p \geq 4$). Thus Hauptvermutung for abelian p-knots ($p \geq 4$) is true. On the other hand, by Siebenmann, "On the homotopy type of compact topological manifolds," *Bull. Amer. math. Soc.* **74** (1968), 738–742, there is a counter-example to the conjecture.

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§2. NOTATIONS AND PRELIMINARY LEMMAS

We continue the notations in the introduction. At first we prove;

LEMMA 2.1. *The subgroup* $TPL(P; Q)$ *is a normal subgroup of* $PL(P; Q)$.

Proof. Let f be an element of $TPL(P; Q)$. Let F be a concordance from f to the identity. Given an element g of $PL(P; Q)$, then we have an element $G = (I \times g^{-1})\ F\ (I \times g)$ of $PL(I \times P;\ I \times Q)$, where $I \times g$ and $I \times g^{-1}$ stand for the maps $(t, x) \mapsto (t, g(x))$, $(t, x) \mapsto (t, g^{-1}(x))$, respectively. Since $G(0, x) = (I \times g^{-1})\ F\ (I \times g)(0, x) = (0, g^{-1}fg(x))$

and $G(1, x), = (1, x)$, it follows that G is a concordance from $g^{-1}fg$ to the identity. Therefore $g^{-1}fg$ belongs to $TPL(P; Q)$, completing the proof.

For our purpose, some other definitions will be needed. For a PL manifold pair (W, M), a subgroup $SPL(W; M)$ of $PL(W; M)$, called the *special* PL homeomorphism group of (W, M), is defined as follows; Each element f of $PL(W)$ induces automorphisms $H^*(f)$ and $H^*(bf)$ of the integral cohomology rings $H^*(W)$ and $H^*(bW)$, where bW stands for the boundary of W. So we have homomorphisms $H^*; PL(W) \to \mathrm{Aut}\, H^*(W)$ and $H^*b\colon PL(W) \to \mathrm{Aut}\, H^*(bW)$ by $H^*f = H^*(f)$ and $H^*bf = H^*(bf)$. We define $SPL(W) = \mathrm{Ker}\, H^* \cap \mathrm{Ker}\, H^*b$, and $SPL(W; M) = PL(W; M) \cap SPL(W)$. Since each element of $TPL(W; M)$ is homotopic to the identity and induces the identity maps of $H^*(W)$ and $H^*(bW)$, it follows that $TPL(W; M)$ is contained in $SPL(W; M)$. The *special* concordance group $C_{SPL}(W; M)$ is defined to be the factor group $SPL(W; M)/TPL(W; M)$

Let D^n be the n-fold cartesian product of the closed interval $[-1, 1]$ endowed with the natural PL structure. Let S^n be the boundary of D^{n+1}.

Let p and q be positive integers. In the following we shall employ the notational convention for manifolds accompanied with $S^p \times S^q$; $T = S^p \times S^q$, $S = S^{p+q+1}$, $S_1 = S^p \times (0)$, $S_2 = (0) \times S^q$, $V_1 = S^p \times D^{q+1}$, $V_2 = D^{p+1} \times S^q$, $\mathbf{T}_1 = (V_1; S_1)$, $\mathbf{T}_2 = (V_2; S_2)$.
We note the identities; $S = V_1 \cup V_2$ and $T = V_1 \cap V_2$. We define two homomorphisms $b : PL(\mathbf{T}_1) \to PL(T)$ and $b' : PL(\mathbf{T}_2) \to PL(T)$ by

$$bg = g \mid T \quad \text{for each element } g \text{ of } PL(\mathbf{T}_1) \quad \text{and}$$
$$b'h = \mid T \quad \text{for each element } h \text{ of } PL(\mathbf{T}_2).$$

LEMMA 2.2. *Given an element g of $PL(\mathbf{T}_1)$, then g belongs to $TPL(\mathbf{T}_1)$ if there is an element F of $SPL(S; S_1)$ such that $F \mid V_1 = g$.*

Proof. Since F is orientation preserving, it follows from the argument in ([9], Lemma 8) that F is isotopic to the identity keeping S_1 fixed. Let G be an isotopy from F to the identity.

Let $V = I \times S_1 \cup (bI) \times V_1$ and $W = I \times S_1 \cup (bI) \times S$.

Then $G(I \times V_1)$ and $I \times V_1$ are regular neighborhoods of V in $I \times S$ mod $(bI) \times V_2$. By the uniqueness of relative regular neighborhoods [5], there is an element H of $PL(I \times S; W)$ such that $HG(I \times V_1) = I \times V_1$. At this point the level preserving property of isotopy breaks down. However, $HG \mid I \times V_1$ is a concordance from the element g to the identity. Thus g belongs to $TPL(\mathbf{T}_1)$, completing the proof.

LEMMA 2.3. $TPL(T) = bPL(\mathbf{T}_1) \cap b'PL(\mathbf{T}_2)$.

Proof. Suppose that an element f of $PL(T)$ belongs to $TPL(T)$. Let F be a concordance from f to the identity. Let $c : I \times T \to V_1$ and $d : I \times T \to V_2$ be PL collars of T in V_1 and V_2, respectively, such that $c(0, x) = \mathrm{d}(0, x) = x$ for any point x in T. Now we define elements g of $PL(\mathbf{T}_1)$ and h of $PL(\mathbf{T}_2)$ by

$$g \mid c(I \times T) = cFc^{-1}, g \mid V_1 - c(I \times T) = \mathrm{id}., \quad \text{and}$$
$$h \mid d(I \times T) = dFd^{-1}, h \mid V_2 - d(I \times T) = id.$$

Then $bg = b'h = f$, which implies that $TPL(T) \subset bPL(\mathbf{T}_1) \cap b'PL(\mathbf{T}_2)$.

Conversely, suppose that for an element f of $PL(T)$ there are elements g of $PL(\mathbf{T}_1)$ and h of $PL(\mathbf{T}_2)$ such that $bg = b'h = f$. We define an element F of $PL(S; S_1)$ by $F | V_1 = g$ and $F | V_2 = h$. Since $F | S_1 \cup S_2 = id.$, it follows that F belongs to $SPL(S; S_1)$. By Lemma 2.2, the element g belongs to $TPL(\mathbf{T}_1)$. The restriction $G | I \times T$ of a concordance G from g to the identity is a concordance from $bg = f$ to the identity. Thus f belongs to $TPL(T)$, completing the proof.

By the last argument in the proof of Lemma 2.3, homomorphisms b and b' induce canonical homomorphisms

$$\beta : C_{PL}(\mathbf{T}_1) \to C_{PL}(T) \quad \text{and} \quad \beta' : C_{PL}(\mathbf{T}_2) \to C_{PL}(T).$$

PROPOSITION 2.4. *The homomorphisms β and β' are monomorphisms.*

Proof. Let g be an element of $PL(\mathbf{T}_1)$ such that bg belongs to $TPL(T)$. Then by Lemma 2.3 there exists an element h of $PL(\mathbf{T}_2)$ such that $bg = b'h$. Hence by Lemma 2.2 the element g belongs to $TPL(\mathbf{T}_1)$. This implies that β is a monomorphism. In the same way we may show that β' is a monomorphism, completing the proof.

Let $\varepsilon_1, \varepsilon_2$ and ε be PL homeomorphisms of $D^{p+1} \times D^{q+1} = D^{p+q+2}$ given by the formulas;

$$\varepsilon_1(x, y_1, y_2, \cdots, y_{q+1}) = (x, -y_1, y_2, \cdots, y_{q+1})$$
$$\varepsilon_2(x_1, x_2, \cdots, x_{p+1}, y) = (-x_1, x_2, \cdots, x_{p+1}, y), \text{ and } \varepsilon(x, y) = (y, x)$$

for any points $x = (x_1, x_2, \cdots, x_{p+1})$ in D^{p+1} and $y = (y_1, y_2, \cdots, y_{q+1})$ in D^{q+1}. For notational convenience, we shall denote restrictions of $\varepsilon_1, \varepsilon_2$ and ε on the appropriate subpolyhedra of D^{p+q+2} by the same notations. Let $H^*(\varepsilon_1) = \varepsilon_1^*$, $H^*(\varepsilon_2) = \varepsilon_2^*$ and $H^*(\varepsilon) = \varepsilon^*$.

LEMMA 2.5. *The group* Aut $H^*(T)$ *is as follows;*

(I) *In case $p \neq q$,* Aut $H^*(T) = Z_2[\varepsilon_1^*] + Z_2[\varepsilon_2^*]$.

(II) *In case $p = q =$ even,* Aut $H^*(T)$ *is a group generated by ε_1^*, ε_2^* and ε^*.*

Proof of (I). Since $p \neq q$, we have Aut $H^*(T) =$ Aut $(H^p(S_1) + H^q(S_2)) =$ Aut $H^p(S_2)$ $+$ Aut $H^q(S_2)$. Moreover, Aut $H^p(S_1) =$ Aut $Z = Z_2$ and Aut $H^q(S_2) =$ Aut $Z = Z_2$ are clearly generated by ε_2^* and ε_1^*, respectively. Thus Aut $H^*(T) = Z_2[\varepsilon_1^*] + Z_2[\varepsilon_2^*]$, since ε_1 and ε_2 generate a subgroup $Z_2[\varepsilon_1] + Z_2[\varepsilon_2]$ of $PL(T)$ which H^* maps isomorphically onto $Z_2[\varepsilon_1^*] + Z_2[\varepsilon_2^*]$, completing the proof.

Proof of (II). Since $p = q =$ even, Aut $H^*(T) =$ Aut $H^p(T) =$ Aut $(Z + Z)$. Let s_1 and s_2 be generators of $H^p(T) = Z + Z$. We may espress each element ξ of Aut $H^p(T)$ by a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ so that

$$\xi s_1 = a s_1 + b s_2 \qquad \text{and} \qquad \xi s_2 = c s_1 + d s_2 .$$

By the invertivility of ξ and by the commutativity of the cup product in $H^*(T)$, we have formulas;

$$(1)\ ad - bc = \pm 1 \quad \text{and} \quad (2)\ ad + bc = \pm 1.$$

On the other hand automorphisms ε_1^*, ε_2^* and ε^* of $H^p(T)$ are expressed by the matrices $\delta_1 = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\delta_2 = -1 \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\delta = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, respectively. Let G be the subgroup of

$GL(2, Z)$ generated by the matrices δ_1, δ_2 and δ. Multiplying either δ_1, δ or $\delta\,\delta_1$ by ξ, if necessary, we may assume that $ad - bc = 1$ and $ad + bc = 1$, i.e., $ad = 1$ and $bc = 0$. Moreover, multiplying $\delta_1\,\delta_2$ by ξ, if necessary, we may assume that $a = d = 1$ and $bc = 0$. If $c = 0$, then $\xi\,\delta\,\xi = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} b & 1+b^2 \\ 1 & b \end{pmatrix}$ belongs to Aut $H^p(T)$. From the condition (2), we have $c = b = 0$. In case $b = 0$, in the quite similar manner, we have the same result. Thus ξ belongs to G. We have, therefore, Aut $H^*(T) = G$, since ε_1, ε_2 and ε generate a subgroup of $PL(T)$ which H^* maps isomorphically onto Aut $H^*(T)$, completing the proof of (II).

Remark. The group G has 8 elements; the identity matrix, δ, δ_1, δ_2, $\delta_1\,\delta_2$, $\delta\,\delta_1$, $\delta\,\delta_2$, and $\delta\,\delta_1\,\delta_2$.

In the sequel we have the following:

COROLLARY 2.6. *Suppose that $p \neq q$ or $p = q =$ even. Then there are splitting exact sequences;*

$$0 \longrightarrow SPL(T) \longrightarrow PL(T) \xrightarrow{H^*} \text{Aut } H^*(T) \longrightarrow 0 \quad \textit{and}$$

$$0 \longrightarrow C_{SPL}(T) \longrightarrow C_{PL}(T) \xrightarrow{H^*} \text{Aut } H^*(T) \longrightarrow 0,$$

that is, each H^* has a right inverse.

§3. DETERMINATION OF THE SET $C_{SPL}(T)$

Let K be a complex. An element f of $PL(|K| \times D^n; |K| \times 0)$ is called to be a prebundle isomorphism of the product prebundle $K \times (D^n, 0)$, if $f(A \times D^n) = A \times D^n$ for any simplex A of K, see [6]. Let Δ_k be the standard k-simplex. The complex consisting of Δ_k and its faces will be denoted by the same notation Δ_k. Recall that a k-simplex of the structural group PR_n of n-prebundles is a prebundle isomorphism of $\Delta_k \times (D_n, 0)$. Let $\{b\Delta_k, SPR_n\}$ be the group of all orientation preserving prebundle isomorphisms of the product prebundle $b\Delta_k \times (D^n, 0)$. Let $b\{\Delta_k, SPR_n\}$ be the subgroup of $\{b\Delta_k, SPR_n\}$ consisting of elements each of which is extendable to a k-simplex of PR_n. Then the homotopy group $\pi_k(PR_n)$ may be interpreted as the factor group $\{b\Delta_{k+1}, SPR_n\}/b\{\Delta_{k+1}, SPR_n\}$ for $k \geq 1$. Identifying $b\Delta_{p+1}$ with S^p we may think of $\{b\Delta_{p+1}, SPR_{q+1}\}$ as a subgroup of $PL(\mathbf{T}_1)$. By Lemma 2.2, we may also think of $b\{\Delta_{p+1}, SPR_{q+1}\}$ as a subgroup of $TPL(\mathbf{T}_1)$.

Thus the inclusion map $\{b\Delta_{p+1}, SPR_{q+1}\} \to PL(\mathbf{T}_1)$ induces a natural homomorphism $\alpha\colon \pi_p(PR_{q+1}) \to C_{PL}(\mathbf{T}_1)$. In the same way we have a natural homomorphism $\alpha'\colon \pi_q(PR_{p+1}) \to C_{PL}(\mathbf{T}_2)$.

LEMMA 3.1. *Let $Z_2[\varepsilon_1]$ and $Z_2[\varepsilon_2]$ be the subgroups of $C_{PL}(\mathbf{T}_1)$ and $C_{PL}(\mathbf{T}_2)$ of order 2 generated by the concordance classes $\{\varepsilon_1\}$ and $\{\varepsilon_2\}$, respectively.*

Then there are splitting exact sequences;

(1) $0 \to \pi_p(PR_{q+1}) \xrightarrow{\alpha} C_{PL}(\mathbf{T}_1) \to Z_2[\varepsilon_1] \to 0,$

(2) $0 \to \pi_q(PR_{q+1}) \xrightarrow{\alpha'} C_{PL}(\mathbf{T}_2) \to Z_2[\varepsilon_2] \to 0.$

Proof. We define the homomorphism $C_{PL}(\mathbf{T}_1) \to Z_2[\varepsilon_1]$ by sending each element $\{f\}$ of $C_{PL}(\mathbf{T}_1)$ to the identity or $\{\varepsilon_1\}$ according as f is orientation preserving or not. If an element f

of $\{b\Delta_{p+1}, SPR_{q+1}\}$ belongs to $TPL(\mathbf{T}_1)$, then by Lemma 2.2 we have an element F of $SPL(S;\ S_1)$ such that $F|\ V_1 = f$. By the join extension, F is extendable to a $(p+1)$-simplex of PR_{q+1}. Hence f belongs to $b\{\Delta_{p+1}, SPR_{q+1}\}$.

Therefore α is injective. Let f be an orientation preserving element of $PL(\mathbf{T}_1)$. We consider two normal prebundles $S^p \times (D^{q+1}, 0)$ and $f(S^p \times (D^{q+1}, 0))$ of S_1 in S. By the uniqueness of normal prebundles of S_1 in S, ([6], Theorem 4.3), we have an element F of $TPL(S;\ S_1)$ such that Ff is a prebundle isomorphism of $b\Delta_{p+1} \times (D^{q+1}, 0)$. Since F is orientation preserving, it follows that Ff belongs to $\{b\Delta_{p+1}, SPR_{q+1}\}$.

By Lemma 2.2, $F|\ V_1$ belongs to $TPL(\mathbf{T}_1)$. Hence f and Ff belong to the same class in $C_{PL}(\mathbf{T}_1)$. Thus $\alpha\{Ff\} = \{f\}$. By the definition of $\pi_p(PR_{q+1})$, an element which represents an element of $\pi_p(PR_{q+1})$ is obviously orientation preserving. This proves the exactness of (1). In the quite similar manner we may prove the exactness of (2), completing the proof.

Combining the existence and uniqueness Theorems of normal PL 2-cell bundles for locally flat PL submanifolds with codimension 2 due to Wall [9] and ([6], Theorem 4.8), we may conclude the following;

PROPOSITION 3.2. *The structural group PR_2 is weakly homotopy equivalent to the structural group πL_2 of PL 2-cell bundles and hence to 0_2.*

Since isomorphism classes of PL 2-cell bundles are completely characterized by the Euler classes, it follows that;

PROPOSITION 3.3. *For every locally flat $PL(p+2, p)$-sphere pair (S^{p+2}, S^p) there is a PL product neighborhood of S^p in S^{p+2}.*

Now we may prove the following extension of the stability Theorem ([6], Theorem 3.3) for the homotopy groups $\pi_k(PR_n)$.

LEMMA 3.4. *There is an isomorphism $w:\ \pi_p(PR_{q+1}, PR_q) \cong \pi_p(S^q)$ for $1 \leq p \leq 2q-3$ and $3 \leq q$, or $p = q = 2$.*

Proof. We define the homomorphism

$$w : \pi_p(PR_{q+1}, PR_q) \to \pi_p(S^q, e)$$

by $w\{f\} = \{p_2 f(\times e)\}$ for each $\{f\}$ in $\pi_p(PR_{q+1}, PR_q)$, where e is a fixed point of S^q, and $p_2 \colon \Delta_p \times S^q \to S^q$ and $(xe) \colon \Delta_p \to \Delta_p \times S^q$ stand for the projection onto the second factor $(x, y) \mapsto y$ and the embedding $x \mapsto (x, e)$, respectively. It is not hard to see that w is well defined. Let $t : \pi_p(0_{q+1}, 0_q) \to \pi_p(PR_{q+1}, PR_q)$ be the homomorphism obtained by triangulating vector bundles into prebundles.

Then the following diagram is commutative:

$$\begin{tikzcd}
\pi_p(PR_{q+1}, PR_q) \arrow[r, "w"] & \pi_p(S^q, e). \\
\pi_p(0_{q+1}, 0_q) \arrow[u, "t"] \arrow[ur, "\cong"'] &
\end{tikzcd}$$

Hence w is surjective. To prove injectivity of w we use the isomorphism $\pi_p(PR_{q+1}, PR_q) \cong \pi_p(\partial PR_{q+1}, \partial_0 PR_{q+1})$ obtained in ([6], Proposition 3.1), where ∂PR_{q+1} and $\partial_0 PR_{q+1}$ stand for the structural groups of S^q and (S^q, e) prebundles, respectively. Suppose that $w\{f\} = 0$ for some $\{f\}$ in $\pi_p(\partial PR_{q+1}, \partial_0 PR_{q+1})$. Then the embedding $f(\ \times e) : (\Delta_p, b\Delta_p) \to (\Delta_p \times S^q, b\Delta_p \times S^q)$ is homotopic to the embedding $(\ \times e)$ keeping $b\Delta_p \times S^q$ fixed.

Suppose that $q \geq 3$ and $2q - 3 \geq p \geq 1$. Since $\Delta_p \times S^q$ is $(q-1)$-connected and Δ_p is contractible, and since $2p - (p+q) = p - q \leq q - 3$ and $q \geq 3$, it follows from Zeeman's embedding Theorem ([15], Theorem 24) that $f(\ \times e)$ and $(\ \times e)$ are ambient isotopic keeping $b\Delta_p \times S^q$ fixed. Hence f belongs to the trivial element of $\pi_p(\partial PR_{q+1}, \partial_0 PR_{q+1})$. Therefore w is injective. Suppose that $p = q = 2$. From the argument above, we note here that $w : \pi_3(PR_4, PR_3) \cong \pi_3(S^3, e)$ and hence $t : \pi_3(0_4, 0_3) \cong \pi_3(PR_4, PR_3)$. Firstly, in order to show that $\pi_2(PR_3) \cong \{0\}$, we observe the following commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_3(PR_4, PR_3) \arrow[r, "\partial_3"] \arrow[d, "w"'] & \pi_2(PR_3) \arrow[r, "i_2"] & \pi_2(PR_4) \\
\cdots \arrow[r] & \pi_3(S^3, e) \arrow[r] & \pi_2(O_3) \arrow[r] \arrow[u, "t"'] & \pi_2(O_4). \arrow[u, "t"', "\cong"]
\end{tikzcd}$$

Since $\pi_2(0_3) \cong \{0\}$, it follows from the commutativity of the diagram that $\partial_3 = 0$. By ([6], Theorem 4.14), we have $\pi_2(0_4) \cong \pi_2(PR_4) \cong \{0\}$. Thus, by the exactness, we have $\pi_2(PR_3) \cong \{0\}$.

Secondly, we show $t : \pi_2(0_3, 0_2) \cong \pi_2(PR_3, PR_2)$. We have from Proposition 3.2, $t : \pi_i(0_2) \cong \pi_i(PR_2)$ for $i = 1, 2$, and from ([6] Theorem 4.14), t: $\pi_1(0_3) \cong \pi_1(PR_3)$. Moreover, as observed above, we have $t : \pi_2(0_3) \cong \pi_2(PR_3)(\cong \{0\})$. It follows from the five Lemma that t: $\pi_2(PR_3, PR_2) \cong \pi_2(0_3, 0_2)$. Therefore, w is an isomorphism, completing the proof.

Let $j_p : \pi_p(PR_{q+1}) \to \pi_p(PR_{q+1}, PR_q)$ be the natural homomorphism.

LEMMA 3.5. *Suppose that $q > p \geq 1$ or $p = q =$ even ≥ 2. Then the composition $j'_p = wj_p$: $\pi_p(PR_{q+1}) \to \pi_p(S^q)$ is a zero map.*

Proof. In case $q > p \geq 1$, since $\pi_p(S^q) = 0$ for $p < q$, it follows obviously that j'_p is a zero map. In case $p = q =$ even, we recall that for an element $\{f\}$ of $\pi_p(PR_{p+1})$, $j'_p\{f\}$ is just the homotopy class of the map $p_2 f(\ \times e) : (\Delta_p, b\Delta_p) \to (S^p, e)$.

On the other hand $\beta\alpha\{f\}$ induces the automorphism $H^p(\beta\alpha\{f\})$ of $H^p(T) = Z + Z$ such that

$$H^p(\beta\alpha\{f\})s_1 = s_1 + a s_2 \qquad \text{and} \qquad H^p(\beta\alpha\{f\})s_1 = s_2,$$

where s_1 and s_2 are generators of $H^p(T) = Z + Z$. Then we have isomorphisms Aut $H^p(T) \cong$ Aut $H_p(T)$ by the Poincaré duality and Aut $H_p(T) \cong$ Aut $\pi_p(T)$ by the Hurewicz isomorphism. By passing such isomorphisms we see that the degree of the map $p_2 f(\ \times e)$ coincides with the integer a up to sign. From the argument in (2) of Lemma 2.5, we have $a = 0$. Thus $j'_p\{f\} = 0$, completing the proof.

COROLLARY 3.6. *Suppose that* $p \neq q$ *or* $p = q = even \geq 2$. *Then* $\{b\Delta_{p+1}, SPR_{q+1}\}$ *is a subgroup of* $SPL(\mathbf{T}_1)$. *In particular,* $\alpha : \pi_p(PR_{q+1}) \to C_{SPL}(\mathbf{T}_1)$ *and* $\alpha' : \pi_q(PR_{p+1}) \to C_{SPL}(\mathbf{T}_2)$ *are isomorphisms.*

Proof. Let f be an element of $\{b\Delta_{p+1}, SPR_{q+1}\}$. If $p \neq q$, then clearly $H^*(bf) = id$. If $p = q =$ even ≥ 2, then from the argument in Lemma 3.5, we have $H^*(bf) = id$. The element f induces the identity of $H^*(V_1)$ and hence belongs to $SPL(\mathbf{T}_1)$. Thus α maps $\pi_p(PR_{q+1})$ into $C_{SPL}(\mathbf{T}_1)$. Since each element of $SPL(\mathbf{T}_1)$ preserves orientation, it follows from Lemma 3.1 that α maps $\pi_p(PR_{q+1})$ onto $C_{SPL}(\mathbf{T}_1)$ and so α is an isomorphism. In the same way we may show that α' is an isomorphism, completing the proof.

COROLLARY 3.7. *Suppose that* $q > p \geq 1$ *or* $p = q = even \geq 2$. *Then the suspension homomorphism* $i_p : \pi_p(PR_q) \to \pi_p(PR_{q+1})$ *is surjective.*

Proof. From Lemmas 3.4 and 3.5, $j'_p : \pi_p(PR_{q+1}) \to \pi_p(PR_{q+1}, PR_q)$ is a zero map. It follows from the exactness of the homotopy sequence for the pair (PR_{q+1}, PR_q) that the homomorphism $i_p : \pi_p(PR_q) \to \pi_p(PR_{q+1})$ is surjective, completing the proof.

Let f be an element of $SPL(T)$. Pasting the boundary T of V_1 onto the boundary T of V_2 by the element f, we have a closed $PL(p + q + 1)$-manifold $S_P = V_1 \underset{f}{\cup} V_2$.

LEMMA 3.8. *Suppose that* $p + q \geq 4$. *Then for any element* f *of* $SPL(T)$, S_f *is a* $PL(p + q + 1)$*-sphere.*

Proof. Since $p \cdot q \geq 2$, from Van Kampen Theorem S_f is simply connected. We note that f induces the identity of $H^*(T)$. Hence observing the Mayer-Vietoris sequence for the identification space $S_f = V_1 \underset{f}{\cup} V_2$ we see that S_f is a homology $(p + q + 1)$-sphere. It follows from the Hurewicz isomorphism Theorem and Whitehead Theorem that S_f is a PL homotopy $(p + q + 1)$-sphere. Then applying the generalized PL Poincaré conjecture of dimension greater than four, we may conclude that S_f is a PL $(p + q + 1)$-sphere, completing the proof.

We shall use the following unknotting Theorem.

THEOREM 3.9 (C. T. C. Wall and J. Levine). *Let* (S^{p+2}, Σ^p) *be a locally flat* $PL(p + 2, p)$*-sphere pair. Suppose that* $p \geq 3$. *Then* (S^{p+2}, Σ^p) *is PL equivalent to the standard PL sphere pair if and only if* $S^{p+2} - \Sigma^p$ *is a homotopy circle.*

For the proof, in case $p \geq 4$, see [13] or [8], and in case $p = 3$, see ([12], p.6, ↓22).

LEMMA 3.10 (H. Sato). *Suppose that* $p \neq q$ *and* $p \cdot q \geq 3$ *or* $p = q = even \geq 2$. *Then the double coset* $b'SPL(\mathbf{T}_2)\backslash SPL(T)/bSPL(\mathbf{T}_1)$ *consists of only one element.*

Proof. This has been first proved by H. Sato for the smooth version in such a case that smooth p-spheres are unknotted in the $(p + q + 1)$-sphere. It suffices to show that for any element f of $SPL(T)$, there are two elements g of $SPL(\mathbf{T}_1)$ and h of $SPL(\mathbf{T}_2)$ such that $b'hfbg = id.$. We may assume that $p \geq q$. let $S_f = V_1 \underset{f}{\cup} V_2$ and let $i_1 : V_1 \to S_f$ and $i_2 : V_2 \to S_f$ be the natural PL embeddings such that $i_2^{-1} i_1 | T = f$. By Lemma 3.8 S_f is a PL $(p + q + 1)$-sphere. In case $q \geq 2$, by Zeeman's unknotting Theorem [14] and by the uniqueness of normal prebundles of the PL embedding $i_1 | S_1$ in S_f over $b\Delta_{p+1}$, we have a PL

homeomorphism $F: S_f \to S$ so that Fi_1 is a prebundle isomorphism of $b\Delta_{p+1} \times (D^{q+1}, 0)$. In case $q = 1$ and $p \geq 3$, since $S_f - i_1(S_1)$ is a homotopy circle, by Theorem 3.9, we have also such a PL homeomorphism F.

Replacing F by $\varepsilon_1 F$, if necessary, we may assume that Fi_1 belongs to $\{b\Delta_{p+1}, SPR_{q+1}\}$ Let $g = (Fi_1)^{-1}$ and $h = Fi_2$. By Corollary 3.6, g belongs to $SPL(\mathbf{T}_1)$. Since bg and $f = i_2^{1} i_1 | T = h^{-1} g^{-1} | T$ belong to $SPL(T)$, it follows that $h|T$ belongs to $SPL(T)$ and hence h belongs to $SPL(V_2)$. Therefore, from the Hurewicz isomorphism Theorem, two PL embeddings $h(\ \times 0)$ and $(\ \times 0): S^q \to V_2$ are homotopic. Since $p + 1 \geq 3$, it follows from Zeeman's embedding Theorem ([15], Theorem 24) that they are ambient isotopic keeping T fixed. So we may assume that h belongs to $SPL(\mathbf{T}_2)$. Thus there are elements g of $SPL(\mathbf{T}_1)$ and h of $SPL(\mathbf{T}_2)$ such that $(b'h)f(bg) = id.$, completing the proof.

We define a function $\gamma: \pi_p(PR_{q+1}) \times \pi_q(PR_{p+1}) \to C_{PL}(T)$ by $\gamma(\{f\}, \{g\}) = (\beta\alpha\{f\})(\beta'\alpha'\{f\})$.

PROPOSITION 3.11 (H. Sato). *The function* $\gamma: \pi_p(PR_{q+1}) \times \pi_q(PR_{p+1}) \to C_{PL}(T)$ *has the following properties*;

(1). *The function* γ *is injective.*

(2). *Suppose that either* $p \neq q$ *and* $p \cdot q \geq 3$ *or* $p = q = even \geq 2$. *Then Image* $\gamma = C_{SPL}(T)$.

(3). *Suppose that* $p \geq q$ *and the suspension homomorphism* $i_p: \pi_p(PR_q) \to \pi_p(PR_{q+1})$ *is surjective. Then the function* γ *is a homomorphism from the direct sum* $\pi_p(PR_{q+1}) + \pi_q(PR_{p+1})$ *into* $C_{PL}(T)$. *In particular, Image* γ *is an abelian subgroup of* $C_{PL}(T)$.

Proof of (1). Suppose that $\gamma(\{f\}, \{g\}) = \gamma(\{h\}, \{k\})$. From the definition of γ, we have $(\beta\alpha\{f\})(\beta'\alpha'\{g\}) = (\beta\alpha\{h\})(\beta'\alpha'\{k\})$, and hence $\beta\alpha\{h^{-1}f\} = \beta'\alpha'\{kg^{-1}\}$.

We note that $bPL(\mathbf{T}_1) \cap TPL(T) \subset bSPL(\mathbf{P}_1)$ and

$$b'PL(\mathbf{T}_2) \cap TPL(T) \subset b'SPL(\mathbf{T}_2).$$

From Lemma 2.2, we obtain $TPL(T) = bPL(\mathbf{T}_1) \cap b'PL(\mathbf{T}_2) = bSPL(\mathbf{T}_1) \cap b'PL(\mathbf{T}_2)$. It follows that $\beta\alpha(\pi_p(PR_{q+1})) \cap \beta'\alpha'(\pi_q(PR_{p+1}))$ has only the trivial element $\{id.\}$. Therefore, $\beta\alpha\{h^{-1}f\} = \beta'\alpha'\{kg^{-1}\} = \{id.\}$. Since α, β, α' and β' are injective, It follows that $\{f\} = \{h\}$ and $\{g\} = \{k\}$, completing the proof.

Proof of (2). From Corollary 3.6 we have Image $\gamma \subset C_{SPL}(T)$. It remains to show that Image $\gamma \supset C_{SPL}(T)$. Let $\{f\}$ be an element of $C_{SPL}(T)$. By Lemma 3.10 there exist elements g of $SPL(\mathbf{T}_1)$ and h of $SPL(\mathbf{T}_2)$ such that $f = bgb'h$. Since g and h are orientation preserving, it follows from Lemma 3.1 that there exist elements φ of $\pi_p(PR_{q+1})$ and ψ of $\pi_q(PR_{p+1})$ such that $\alpha\varphi = \{g\}$ and $\alpha'\psi = \{h\}$. Hence $\gamma(\varphi, \psi) = (\beta\alpha\varphi)(\beta'\alpha'\psi) = \beta\{g\}\beta'\{h\} = \{bg\}\{b'h\} = \{f\}$. Therefore $C_{SPL}(T) \subset$ Image γ, completing the proof.

Proof of (3). It suffices to show that for any pair (φ, ψ) of $\pi_p(PR_{q+1}) \times \pi_q(PR_{p+1})$, $(\beta\alpha\varphi)(\beta'\alpha'\psi) = (\beta'\alpha'\psi)(\beta\alpha\varphi)$. Since $p \geq q$, by Corollary 3.7 the suspension homomorphism $i_q: \pi_q(PR_p) \to \pi_q(PR_{p+1})$ is also surjective. Hence $\alpha\varphi$ is represented by an element $f \times D$ for some element f of $PL(S^p \times D^q; E_1)$ and $\alpha'\psi$ is represented by an element $D \times g$ for some element g of $PL(D^p \times S^q; E_2)$, where $E_1 = S^p \times (0) \cup (S^p - D^p) \times D^q$, $E_2 = (0) \times S^q \cup D^p \times (S^q - D^p)$ and $f \times D$ and $D \times g$ stand for the maps $(x, u) \mapsto (f(x), u)$ and (u, x)

$\rightarrow (u, g(x))$ respectively. Then the element $(\beta'\alpha'\psi)^{-1}(\beta\alpha\varphi)^{-1}(\beta'\alpha'\psi)(\beta\alpha\varphi)$ is represented by

$$\begin{aligned} h &= b'(D \times g^{-1})b(f^{-1} \times D)b'(D \times g)b(f \times D) \\ &= (D \times g^{-1})(f^{-1} \times D)(D \times g)(f \times D) \,|\, T. \end{aligned}$$

Since h is supported by the ball $(1) \times D^p \times D^q \times (1)$, it follows that h is isotopic to the identity. W have, therefore, $(\beta'\alpha'\psi)(\beta\alpha\varphi) = (\beta\alpha\varphi)(\beta'\alpha'\psi)$, completing the proof.

§4. PROOF OF THEOREMS

Proof of Theorem A. Since $p \neq q$, it follows from Lemma 2.5, (I) and Corollary 2.6 that there is a one-to-one correspondence between $C_{PL}(T)$ and $C_{SPL}(T) \times (Z_2 + Z_2)$. On the other hand, by Proposition 3.11 there is a bijection $\gamma : \pi_p(PR_{q+1}) \times \pi_q(PR_{p+1}) \rightarrow C_{SPL}(T)$, completing the proof.

Proof of Theorem B. Since $p = q = \text{even} \geq 2$, it follows from Lemma 2.5, (II) and Corollary 2.6 that there is a one-to-one correspondence between $C_{PL}(T)$ and $C_{SPL}(T) \times G$. By Corollary 3.7, $i_p : \pi_p(PR_p) \rightarrow \pi_p(PR_{p+1})$ is surjective. It follows from (2) and (3) in Proposition 3.11 that the function γ is an isomorphism from $\pi_p(PR_{p+1}) + \pi_p(PR_{p+1})$ onto $C_{SPL}(T)$, completing the proof.

Proof of Theorem C. Since $p \neq 1$, it follows from Lemma 2.5 and Corollary 2.6 that there is a one-to-one correspondence between $C_{PL}(T)$ and $C_{SPL}(T) \times (Z_2 + Z_2)$. By Proposition 3.2, $\pi_p(PR_2) \cong \{0\}$ for $p \geq 2$. Hence, in case $p \geq 3$, by (2) and (3) in Proposition 3.11, and in case $p = 2$, by ([4], Theorem 13.2) there is an isomorphism $\gamma : Z_2 \rightarrow C_{SPL}(T)$. Let δ be an element of $C_{PL}(T)$ corresponding to an element of $Z_2 + Z_2 \cong \text{Aut } H^*(T)$. Let φ be a generator of $C_{SPL}(T) \cong Z_2$. Since $H^*(\delta^{-1}\varphi\delta) = id.$, $\delta^{-1}\varphi\delta$ belongs to $C_{SPL}(T) \cong Z_2$. Hence we have $\delta^{-1}\varphi\,\delta = \varphi$, or $\varphi\delta = \delta\varphi$. We have therefore an isomorphism $Z_2 + Z_2 + Z_2 \cong C_{PL}(T)$, completing the proof.

Proof of Theorem D. Let $G(S^p \times S^q)$ be the H-space of homotopy equivalences of $S^p \times S^q$ onto itself. Let G_n be the H-space of homotopy equivalences of S^{n-1} onto itself. Then we obtain a homomorphism $\varphi : \pi_q(G^2{}_{p+1}) \rightarrow \pi_0(G(S^p \times S^q))$ as follows; If a map $f : S^q \rightarrow G_{p+1}$ represents an element $\{f\}$ of $\pi_q(G_{p+1})$, then the corresponding element $\varphi\{f\}$ is defined to be the homotopy class of a map $f' : S^p \times S^q \rightarrow S^p \times S^q$ which is induced from f by $f'(x, y) = (f(y)(x), y)$ for x in S^p and y in S^q. It is not hard to see that φ is a monomorphism (refer the Proposition 2.4). We consider the homomorphism $i : C_{PL}(S^p \times S^q) \rightarrow \pi_0(G(S^p \times S^q))$ induced from the inclusion map $PL(S^p \times S^q) \subset H(S^p \times S^q)$. Suppose that $p + 2 \leq q$. Consider the following commutative diagram;

$$\begin{tikzcd}
\pi_p(PR_{q+1}) \arrow[r, "\beta\alpha"] & C_{SPL}(T) \arrow[dr, "i'"] & \\
& & \pi_0(G(T)), \\
\pi_p(O_{q+1}) \arrow[uu, "\iota"] \arrow[r, "J_p"] & \pi_p(G_{q+1}) \arrow[ur, "\varphi"] &
\end{tikzcd}$$

where $t\colon \pi_p(0_{q+1}) \to \pi_p(PR_{q+1})$ is the monomorphism in ([6], Theorem 4.14) obtained from Hirsch-Mazur's exact sequence, $i' = i \mid C_{SPL}(T)$, and $J_p\colon \pi_p(0_{q+1}) \to \pi_p(G_{q+1}) \cong \pi_{p+q}(S^q)$ is essentially the J-homomorphism. Now $\pi_{p+q}(S^q)$ is a stable group and hence finite, but for $p \equiv 3 \pmod 4$ $\pi_p(0_{q+1}) \cong Z$ hence Ker $J_p \neq (0)$. Let $x = \text{Ker } J_p$ be non-zero and let $\xi = \beta\alpha t(x)$. Then $i'(\xi) = 0$ since $J_p(x) = 0$ but $\beta\alpha t$ is injective.

So if f represents ξ, f is not concordant to the identity but is homotopic to it.

Therefore f is not isotopic to the identity. Thus for $q \geq p + 2$ and $p \equiv 3 \pmod 4$ $T = S^p \times S^q$ is self-knotted, completing the proof.

Proof of Theorem E. We note that the inclusion map $0_{p+1} \subset G_{p+1}$ induces an isomorphism $\pi_1(0_{p+1}) \cong \pi_1(G_{p+1})$. From ([6], Theorem 4.14) and Proposition 3.11, we have isomorphisms $\pi_1(0_{p+1}) \cong \pi_1(PR_{p+1}) = C_{SPL}(S^p \times S^1)$ such that the following diagram commutes;

$$\begin{array}{ccc} \pi_1(O_{p+1}) & \xrightarrow{\cong} & \pi_1(G_{p+1}) \\ \downarrow \wr\Vert & & \downarrow \varphi \\ C_{SPL}(S^p \times S^1) & \xrightarrow{i'} & \pi_0(G(S^p \times S^1)), \end{array} \quad \text{where } i' = i \mid C_{SPL}(S^p \times S^1).$$

Since φ is a monomorphism, it follows that i' is a monomorphism. The kernel of i is clearly contained in $C_{SPL}(S^p \times S^1)$. Thus i is a monomorphism, completing the proof.

Proof of Theorem F. Let (S^{p+2}, S_p^1) and (S^{p+2}, S_1^p) be two locally flat PL $(p+2, p)$-sphere pairs. Let N_i be a regular neighborhood of S_i^p in S^{p+2} for each $i = 1, 2$. Let E_i be an exterior $S^{p+2} - N_i$ for each $i = 1, 2$. Suppose that $p \geq 2$ and E_1, E_2 are PL homeomorphic to a PL manifold E with boundary $S^p \times S^1$. Let $g_i\colon E \to E_i$ be a PL homeomorphism for each $i = 1, 2$. By Proposition 3.3 there is a PL homeomorphism $f_i\colon (S^p \times D^2, S^p \times (0)) \to (S^{p+2}, S_i^p)$ for each i $= 1, 2$. Let $k_i = g^{-1}f_i \mid S^p \times S^1$ for each i $= 1,2$. We note that the PL homeomorphisms ε_1 and ε_2 are considered as elements of $PL(S^p \times D^2)$ such that $\varepsilon_i(S^p \times (0)) = S^p \times (0)$, i $= 1, 2$. By Theorem C, replacing f_i by either $f_i\varepsilon_1$, $f_i\varepsilon_2$ or $f_i\varepsilon_1\varepsilon_2$ if necessary, we may assume that the concordance class of k_i belongs to $C_{SPL}(S^p \times S^1) \cong Z_2$ for each $i = 1, 2$. If k_1 and k_2 belong to the same class in $C_{SPL}(S^p \times S^1)$, then $k_2^{-1}k_1$ is concordant to the identity. Hence we may extend $k_2^{-1}k_1$ to an element k of $PL(S^p \times D^2;$ $S^p \times (0))$ as in the proof of Lemma 2.3. Then we have a PL homeomorphism $h\colon (S^{p+2}, S_1^p) \to (S^{p+2}, S_2^p)$ by setting $h \mid E_1 = g_2 g^{-1}$ and $h \mid N_1 = f_2 k f_1^{-1}$. Thus (S^{p+2}, S_1^p) and (S^{p+2}, S_2^p) are equivalent. Since $C_{SPL}(S^p \times S^1) \cong Z_2$, it follows that there are at most two inequivalent PL locally flat $(p+2, p)$-sphere pairs with a given exterior E, completing the proof.

§5. PROOF OF THEOREM G

In order to prove Theorem G we need the following Lemma;

LEMMA 5.1. *Let (W, M) and (W', M') be two PL $(m+n, m)$-manifold pairs consisting of closed PL manifolds. Given a homeomorphism $g\colon (W, M) \to (W', M')$, then there exist regular neighborhoods N of M in W and N' of M' in W' and a map $f\colon (W, N, M) \to (W', N', M')$ such that $f \mid E\colon (E, T) \to (E', T')$ and $f \mid N\colon (N, T, M) \to (N', T', M')$ are* *homotopy equivalences, where* $E = W - N$, $E' = W' - N'$, $T = bE = bN$ *and* $T' = bE' = bN'$.

Proof. We may take regular neighborhoods N_1, N_2 and N_3 of M in W and N_1, N_2 and N_3 of M' in W' so that

$$g(N_1) \supset N_1', g^{-1}(\text{Int } N_1') \supset N_2, g(\text{Int } N_2) \supset N_2',$$
$$g^{-1}(\text{Int } N_2') \supset N_3, \text{ and } g(N_3) \supset N_3'. \text{ Let } A = \overline{N_2\text{-}N_3} \text{ and } A' = \overline{N_1'\text{-}N_2'}.$$

By the regular neighborhood annulus Theorem [4], we may identify A and A' with $I \times bN_3$ and $I \times bN_2'$, respectively, so that $bN_3 = (0) \times bN_3$ and $bN_2' = (0) \times bN_2'$. Then there exists a positive number ε such that

$$g([1-\varepsilon, 1] \times bN_3) \subset (\varepsilon, 1-\varepsilon) \times bN_2' \quad \text{and}$$
$$g([0, \varepsilon] \times bN_3) \subset \text{Int } N_2' - N_3'.$$

We define two maps $\alpha\colon A \to A$ and $\alpha'\colon A' \to A'$ by

$$\alpha(t, x) = (\varepsilon^{-1}(1-\varepsilon)t, x), \alpha'(t, y) = (t, y) \text{ for } (t, x, y) \text{ in } [0, \varepsilon] \times bN_3 \times bN_2',$$
$$\alpha(t, x) = (1-\varepsilon, x), \alpha'(t, y) = (\varepsilon, y) \text{ for } (t, x, y) \text{ in } [\varepsilon, 1-\varepsilon] \times bN_3 \times bN_2', \quad \text{and}$$
$$\alpha(t, x) = (t, x), \alpha'(t, y) = (\varepsilon^{-1}((1-\varepsilon)t + 2\varepsilon - 1), y) \text{ for } (t, x, y) \text{ in } [1-\varepsilon, 1] \times bN_3\ bN \times {}_2'.$$

Since $\alpha \mid bA = id.$ and $\alpha' \mid bA' = id.$, we may define two maps $\beta\colon (W, M) \to (W, M)$ and $\beta'\colon (W', M') \to (W', M')$ by

$$\beta \mid \overline{W\text{-}A} = id., \beta \mid A = \alpha \qquad \text{and}$$
$$\beta' \mid \overline{W'\text{-}A'} = id., \beta' \mid A' = \alpha'.$$

Let $N = N_3 \cup [0, 1-\varepsilon] \times bN_3$, $N' = N_2' \cup [0, 1-\varepsilon'] \times bN_2$, $f = \beta' g$ and $f' = \beta' g^{-1}$. Then N and N' are clearly regular neighborhoods of M and M' in W and W', respectively. We shall show in the following that $f\colon (W, M) \to (W', M')$ satisfies the required condition. We note that $f(N) = N'$, $f'f \mid \overline{W\text{-}N_1} = id.$, $ff' \mid \overline{W'\text{-}N_1} = id.$, $f'f \mid N_3 = id.$, and $f \cdot f' \mid N_3 = id.$. Since by the regular neighborhood annulus Theorem $\overline{N_1\text{-}N}$ and $\overline{N'\text{-}N'}$ are identified with $I \times bN$ and $I \times bN'$, respectively, and since $f'f \mid I \times bN$ and $ff' \mid I \times bN'$ are considered as homotopies from $f'f \mid bN$ and $ff' \mid bN'$ to the identity maps, respectively, it follows that $f'f \mid (E, bE)$ and $f' \mid (E', bE')$ are homotopic to the identity maps keeping $\overline{W\text{-}N_1}$ and $\overline{W'\text{-}N_1}$ fixed, where $E = W - N$ and $E' = W'\text{-}N'$. In the same way, we may show that $f'f \mid (N, bN)$ and $ff' \mid (N', bN')$ are homotopic to the identity maps keeping N_3 and N_3 fixed, completing the proof.

Proof of Theorem G. For notational convenience we put $S^{p+2} = S$, $S_1^p = S_1$ and $S_2^p = S_2$. By Lemma 5.1, we have two regular neighborhoods N_1 of S_1 in S and N_2 of S_2 in S and a map $f\colon (S/N_1,, S_1) \to (S, N_2, S_2)$ so that $f_1\colon (E_1, T_1) \to (E_2, T_2)$ and $f_2\colon (N_1, T_1, S_1) \to (N_2, T_2, S_2)$ are homotopy equivalences, where $E_i = \overline{S\text{-}N_i}$, $T_i = bE_i = bN_i$, for $i = 1, 2$, and $f_1 = f \mid E_1$ and $f_2 = f \mid N_1$. Since by the uniqueness of regular neighborhoods the *PL* homeomorphism classes of the exteriors E_1 and E_2 are unique, it follows from the assumption that there exists a *PL* homeomorphism $h_1\colon (E_1, T_1) \to (E_2, T_2)$ which is homotopic to $f_1\colon (E_1, T_1) \to (E_2, T_2)$. From Proposition 3.3, we have *PL* homeomorphisms $c_i\colon (S_i \times D^2, S_i \times (0))$

$\to (N_i, S_i)$, $i = 1, 2$ such that $c_i(x, 0) = x$ for x in S_i, $i = 1, 2$. Then $c_2^{-1} f c_1: (S_1 \times D^2, S_1 \times (0)) \to (S_2 \times D^2, S_2 \times (0))$ is a homotopy equivalence. By the covering homotopy Theorem, the map $c_2^{-1} f c_1$ is homotopic to a fiber preserving map $g_1: (S_1 \times D^2, S_1 \times (0)) \to (S_2 x D^2, S_2 \times (0))$. Since g_1 is a homotopy equivalence and since $\pi_p(G_2) \cong \pi_p(0_2) \cong \pi_p(\pi L_2)$ for all p, it follows that g_1 is fiberwise homotopic to a PL homeomorphism g_2: $(S_1 \times D^2, S_1 x(0)) \to (S_2 \times D^2, S_2 \times (0))$, where πL_2 stands for the c.s.s.) structural group of PL 2 cell bundle.

We put $h_2 = c_2 g_2 c_1^{-1}$.

Since $h_1 | T_1: T_1 \to T_2$ and $h_2 | T_1: T_1 \to T_2$ are homotopic to the homotopy equivalence $f | T_1: T_1 \to T_2$, it follows that $h_2^{-1} h_1 | T_1: T_1 \to T_1$ is homotopic to the identity map. From Corollary to Theorem C, $h_2^{-1} h_1 | T_1$ is concordant to the identity map. Hence we have a PL homeomorphism $h_3: (N_1, S_1) \to (N_2, S_2)$ such that $h_3 | T_1 = h_2^{-1} h_1 | T_1$. Thus we may define a PL homeomorphism $h: (S, S_1) \to (S, S_2)$ by setting $h | E_1 = h_1$ and $h | N_1 = h_2 h_3$, completing the proof.

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